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A possible solution for the non-existence of time

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Abstract. A possible solution for the problem of non-existence of universal time is given by utilizing Gödel's incompleteness theorem [2].

In a recent book [1], Barbour presented a thought that time is an illusion, by noting that Wheeler-DeWitt equation yields the non-existence of time, whereas time around us seems to be flowing. However, he does not appear to have a definite idea or formal way to actualize his thought. In the present note I present a concrete way to resolve the problem of the non-existence of time, which is partly a reminiscence of my works [4], [5], [6], [7].

1. Time seems not to exist

According to equation (5.13) in Hartle [3], the non-existence of time would be expressed by an equation:

$$H\Psi = 0. \tag{1}$$

Here Ψ is the "state" of the universe belonging to a suitable Hilbert space $\mathcal{H}$, and H denotes the total Hamiltonian of the universe defined in $\mathcal{H}$. This equation implies that there is no global time of the universe, as the state Ψ of the universe is an eigenstate for the total Hamiltonian H, and therefore does not change. One might think that this implies the non-existence of local time because any part of the universe is described by a part of Ψ. Then we have no time, in contradiction with our observations. This is a restatement of the problem of time, which is a general problem to identify a time coordinate while preserving the diffeomorphism invariance of General Relativity. In fact, equation (1) follows if one assumes the existence of a preferred foliating family of spacelike surfaces in spacetime (see section 5 of [3]).

We give a solution in the paper to this problem that on the level of the total universe, time does not exist, but on the local level of our neighborhood, time does exist.

2. Gödel's theorem

Our starting point is the incompleteness theorem proved by Gödel [2]. It states that any consistent formal theory that can describe number theory includes an infinite number of undecidable propositions. The physical world includes at least natural numbers, and it is described by a system of words, which can be translated into a formal physics theory. The theory of physics, if consistent, therefore includes an undecidable proposition, i.e. a proposition whose correctness cannot be known by human beings until one finds a phenomenon or observation that supports the proposition or denies the proposition. Such propositions exist infinitely according to Gödel's theorem. Thus human beings, or any other finite entity, will never be able to reach a "final" theory that can express the totality of the phenomena in the universe.

Thus we have to assume that any human observer sees a part or subsystem L of the universe and never gets the total Hamiltonian H in (1) by his observation. Here the total Hamiltonian H is an *ideal* Hamiltonian that might be gotten by "God." In other words, a consequence from Gödel's theorem is that the Hamiltonian that an observer assumes with his observable universe is a part H_L of H. Stating explicitly, the consequence from Gödel's theorem is the following proposition

$$H = H_L + I + H_E, \quad H_E \neq 0, \tag{2}$$

where H_E is an unknown Hamiltonian describing the system E exterior to the realm of the observer, whose existence, i.e. $H_E \neq 0$, is assured by Gödel's theorem. This unknown system E includes all that is unknown to the observer. E.g., it might contain particles which exist near us but have not been discovered yet, or are unobservable for some reason at the time of observation. The term I is an unknown interaction between the observed system L and the unknown system E. Since the exterior system E is assured to exist by Gödel's theorem, the interaction I does not vanish: In fact assume I vanishes. Then the observed system L and the exterior system E do not interact, which is the same as that the exterior system E does not exist for the observer. This contradicts that the observer is able to construct a proposition by Gödel's procedure (see section 5 and [2]) that proves E exists. By the same reason, I is not a constant operator:

$$I \neq \text{constant operator.} \tag{3}$$

For suppose it is a constant operator. Then the systems L and E do not change no matter how far or how near they are located because the interaction between L and E is a constant operator. This is the same situation as that the interaction does not exist, thus reduces to the case $I = 0$ above.

We now arrive at the following observation: For an observer, the observable universe is a part L of the total universe and it looks as though it follows the Hamiltonian H_L, not following the total Hamiltonian H. And the state of the system L is described by a part $\Psi(\cdot, y)$ of the state Ψ of the total universe, where y is an unknown coordinate of system L inside the total universe, and $\cdot$ is the variable controllable by the observer, which we will denote by x.

3. Local Time Exists

Assume now, as is usually expected, that there is no local time of L, i.e. that the state $\Psi(x,y)$ is an eigenstate of the local Hamiltonian H_L for some $y = y_0$ and a real number μ:

$$H_L\Psi(x,y_0) = \mu\Psi(x,y_0). \tag{4}$$

Then from (1), (2) and (4) follows that

$$\begin{aligned} 0 = H\Psi(x,y_0) &= H_L\Psi(x,y_0) + I(x,y_0)\Psi(x,y_0) + H_E\Psi(x,y_0) \\ &= (\mu + I(x,y_0))\Psi(x,y_0) + H_E\Psi(x,y_0). \end{aligned} \tag{5}$$

Here x varies over the possible positions of the particles inside L. On the other hand, since H_E is the Hamiltonian describing the system E exterior to L, it does not affect the variable x and acts only on the variable y. Thus $H_E\Psi(x,y_0)$ varies as a bare function $\Psi(x,y_0)$ insofar as the variable x is concerned. Equation (5) is now written: For all x

$$H_E\Psi(x,y_0) = -(\mu + I(x,y_0))\Psi(x,y_0). \tag{6}$$

As we have seen in (3), the interaction I is not a constant operator and varies when x varies[†], whereas the action of H_E on Ψ does not. Thus there is a nonempty set of points x_0 where $H_E\Psi(x_0,y_0)$ and $-(\mu + I(x_0,y_0))\Psi(x_0,y_0)$ are different, and (6) does not hold at such points x_0. If I is assumed to be continuous in the variables x and y, these points x_0 constitutes a set of positive measure. This then implies that our assumption (4) is wrong. Thus a subsystem L of the universe cannot be a bound state with respect to the observer's Hamiltonian H_L. This means that the system L is observed as a non-stationary system, therefore there must be observed a motion inside the system L. This proves that the "time" of the local system L *exists for the observer* as a measure of motion, whereas the total universe is stationary and does not have "time."

4. A refined argument

To show the argument in section 3 more explicitly, we consider a simple case of

$$H = \frac{1}{2}\sum_{k=1}^{N} h^{ab}(X_k)p_{ka}p_{kb} + V(X).$$

Here N ($1 \le N \le \infty$) is the number of particles in the universe, h^{ab} is a three-metric, $X_k \in R^3$ is the position of the k-th particle, p_{ka} is a functional derivative corresponding to momenta of the k-th particle, and $V(X)$ is a potential. The configuration

[†]Note that Gödel's theorem applies to any fixed $y = y_0$ in (3). Namely, for any position y_0 of the system L in the universe, the observer must be able to know that the exterior system E exists because Gödel's theorem is a universal statement valid throughout the universe. Hence $I(x,y_0)$ is not a constant operator with respect to x for any fixed y_0.

$X = (X_1, X_2, \cdots, X_N)$ of total particles is decomposed as $X = (x, y)$ accordingly to if the k-th particle is inside L or not, i.e. if the k-th particle is in L, X_k is a component of x and if not it is that of y. H is decomposed as follows:

$$H = H_L + I + H_E.$$

Here H_L is the Hamiltonian of a subsystem L that acts only on x, H_E is the Hamiltonian describing the exterior E of L that acts only on y, and $I = I(x, y)$ is the interaction between the systems L and E. Note that H_L and H_E commute.

Theorem. Let P denote the eigenprojection onto the space of all bound states of H. Let P_L be the eigenprojection for H_L. Then we have

$$(1 - P_L)P \neq 0, \tag{7}$$

unless the interaction $I = I(x, y)$ is a constant with respect to x for any y.

Remark. In the context of the former part, the theorem implies the following:

$$(1 - P_L)P\mathcal{H} \neq \{0\},$$

where $\mathcal{H}$ is a Hilbert space consisting of all possible states Ψ of the total universe. This relation implies that there is a vector $\Psi \neq 0$ in $\mathcal{H}$ which satisfies $H\Psi = \lambda\Psi$ for a real number λ while $H_L\Phi \neq \mu\Phi$ for any real number μ, where $\Phi = \Psi(\cdot, y)$ is a state vector of the subsystem L with an appropriate choice of the position y of the subsystem.

Proof of the theorem. Assume that (7) is incorrect. Then we have

$$P_L P = P.$$

Taking the adjoint operators on the both sides, we then have

$$PP_L = P.$$

Thus $[P_L, P] = P_L P - PP_L = 0$. But in generic this does not hold because

$$[H_L, H] = [H_L, H_L + I + H_E] = [H_L, I] \neq 0,$$

unless $I(x, y)$ is equal to a constant with respect to x. Q.E.D.

5. Conclusion

Gödel's proof of the incompleteness theorem relies on the following type of proposition P insofar as concerned with the meaning:

$$P \equiv \text{“}P \text{ cannot be proved.”} \tag{8}$$

Then if P is provable it contradicts P itself, and if P is not provable, P is correct and seems to be provable. Both cases lead to contradiction, which makes this kind of proposition undecidable in a given consistent formal theory.

This proposition reminds us of the following type of self-referential statement:

$$\text{A person } P \text{ says ``I am telling a lie.''} \tag{9}$$

The above statement and proposition P in (8) are non-diagonal statements in the sense that both deny themselves. Namely the core of Gödel's theorem is in proving the existence of non-diagonal "elements" (i.e. propositions) in any formal theory that includes number theory. Assigning the so-called Gödel number to each proposition in number theory, Gödel constructs such propositions in number theory by a diagonal argument, which shows that any consistent formal theory has a region exterior to the knowable world.

On the other hand, what we have deduced from Gödel's theorem in section 2 is that the interaction term I is not a constant operator. Moreover the argument there implies that I is not diagonalizable in the following decomposition of the Hilbert space $\mathcal{H}$:

$$\mathcal{H} = \int^{\oplus} \mathcal{H}_L(\lambda) d\lambda \otimes \int^{\oplus} \mathcal{H}_E(\mu) d\mu, \tag{10}$$

where the first factor on the RHS is the decomposition of $\mathcal{H}$ with respect to the spectral representation of H_L, and the second is the one with respect to that of H_E. In this decomposition, $H_0 = H_L + H_E$ is decomposed as a diagonal operator:

$$H_0 = H_L \otimes I_E + I_L \otimes H_E = \int^{\oplus} \lambda d\lambda \otimes I_E + I_L \otimes \int^{\oplus} \mu d\mu,$$

where I_L and I_E denote identity operators in respective factors in (10). To see that I is not diagonalizable in the decomposition (10), assume contrarily that I is diagonalizable with respect to (10). Then by spectral theory of selfadjoint operators, I is decomposed as $I = f(H_L) \otimes I_E + I_L \otimes g(H_E)$ for some functions $f(H_L)$ and $g(H_E)$ of H_L and H_E. Thus the total Hamiltonian H is also diagonalizable and written as:

$$H = H_0 + I = (H_L + f(H_L)) \otimes I_E + I_L \otimes (H_E + g(H_E)).$$

Namely the total Hamiltonian H is decomposed into a sum of mutually independent operators in the decomposition of the total system into the observable and unobservable systems L and E. This means that there are no interactions between L and E, contradicting Gödel's theorem as in section 2. Therefore I is not diagonalizable with respect to the direct integral decomposition (10) of the space $\mathcal{H}$.

Now a consequence of Gödel's theorem in the context of the decomposition of the total universe into observable and unobservable systems L and E is the following:

> In the spectral decomposition (10) of $\mathcal{H}$ with respect to a decomposition of the total system into the observable and unobservable ones, I is non-diagonalizable. In particular so is the total Hamiltonian $H = H_L + I + H_E$.

Namely Gödel's theorem yields the existence of non-diagonal elements in the spectral representation of H with respect to the decomposition of the universe into observable and unobservable systems. The existence of non-diagonal elements in this decomposition is the cause that the observable state $\Psi(\cdot, y)$ is not a stationary state and local time arises, and that decomposition is inevitable by the existence of the region unknowable to human beings.

From the standpoint of the person P in (9), his universe needs to proceed to the future for his statement to be decided true or false; the decision of which requires his system to have infinite "time." This is due to the fact that his self-contradictory statement does not give him satisfaction in his own world and forces him to go out to the region exterior to his universe. Likewise, the interaction I in the decomposition above forces the observer to anticipate the existence of a region exterior to his knowledge. In both cases the unbalance caused by the existence of an exterior region yields time. In other words, time is an indefinite desire to reach the balance that only the universe has.

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